3.9 Fractional Step Methods

An approximation of multidimensional problems similar to ADI (or, in general, approximate factorization schemes) is the method of fractional step. This method splits the multidimensional equation into a series of one-space dimensional equations and solves them sequentially. For the two-dimensional model equation

$$\frac{\partial u}{\partial t} = \alpha \left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right]$$

the method provides the following finite difference equations: (Note that the Crank-Nicolson scheme is used.)

$$\frac{u_{i,j}^{n+\frac{1}{2}} - u_{i,j}^n}{\frac{\Delta t}{2}} = \alpha \frac{1}{2} \left[\frac{u_{i+1,j}^{n+\frac{1}{2}} - 2u_{i,j}^{n+\frac{1}{2}} + u_{i-1,j}^{n+\frac{1}{2}}}{(\Delta x)^2} + \frac{u_{i+1,j}^n - 2u_{i,j}^n + u_{i-1,j}^n}{(\Delta x)^2} \right]$$

and

$$\frac{u_{i,j}^{n+1} - u_{i,j}^{n+\frac{1}{2}}}{\frac{\Delta t}{2}} = \alpha \frac{1}{2} \left[\frac{u_{i,j+1}^{n+1} - 2u_{i,j}^{n+1} + u_{i,j-1}^{n+1}}{(\Delta y)^2} + \frac{u_{i,j+1}^{n+\frac{1}{2}} - 2u_{i,j}^{n+\frac{1}{2}} + u_{i,j-1}^{n+\frac{1}{2}}}{(\Delta y)^2} \right]$$

The scheme is unconditionally stable and is of order $[(\Delta t)^2, (\Delta x)^2, (\Delta y)^2]$.

3.10 Extension to Three-Space Dimensions

The ADI method just investigated for the unsteady two-space dimensional parabolic equation can be extended to three-space dimensions, which is accomplished by considering time intervals of n, $n + \frac{1}{3}$, $n + \frac{2}{3}$, and $n + 1$. The resulting equations for the model equation

$$\frac{\partial u}{\partial t} = \alpha \left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right] \tag{3-35}$$

are:

$$\frac{u_{i,j,k}^{n+\frac{1}{3}} - u_{i,j,k}^n}{\frac{\Delta t}{3}} = \alpha \left[\frac{\delta_x^2 u_{i,j,k}^{n+\frac{1}{3}}}{(\Delta x)^2} + \frac{\delta_y^2 u_{i,j,k}^n}{(\Delta y)^2} + \frac{\delta_z^2 u_{i,j,k}^n}{(\Delta z)^2} \right]$$

$$\frac{u_{i,j,k}^{n+\frac{2}{3}} - u_{i,j,k}^{n+\frac{1}{3}}}{\frac{\Delta t}{3}} = \alpha \left[\frac{\delta_x^2 u_{i,j,k}^{n+\frac{1}{3}}}{(\Delta x)^2} + \frac{\delta_y^2 u_{i,j,k}^{n+\frac{2}{3}}}{(\Delta y)^2} + \frac{\delta_z^2 u_{i,j,k}^{n+\frac{1}{3}}}{(\Delta z)^2} \right]$$

and

$$\frac{u_{i,j,k}^{n+1} - u_{i,j,k}^{n+\frac{2}{3}}}{\frac{\Delta t}{3}} = \alpha \left[\frac{\delta_x^2 u_{i,j,k}^{n+\frac{2}{3}}}{(\Delta x)^2} + \frac{\delta_y^2 u_{i,j,k}^{n+\frac{2}{3}}}{(\Delta y)^2} + \frac{\delta_z^2 u_{i,j,k}^{n+1}}{(\Delta z)^2} \right]$$

The method is of order $[(\Delta t), (\Delta x)^2, (\Delta y)^2, (\Delta z)^2]$ and is only conditionally stable with the requirement of $(d_x + d_y + d_z) \leq (3/2)$. As a result of this requirement, the method is not very attractive. A method that is unconditionally stable and is second-order accurate uses the Crank-Nicolson scheme. The finite difference equations of the model Equation (3-35) are

$$\frac{u^{*}_{i,j,k} - u^{n}_{i,j,k}}{\Delta t} = \alpha\left[\frac{1}{2}\frac{\delta^2_x u^{*}_{i,j,k} + \delta^2_x u^{n}_{i,j,k}}{(\Delta x)^2} + \frac{\delta^2_y u^{n}_{i,j,k}}{(\Delta y)^2} + \frac{\delta^2_z u^{n}_{i,j,k}}{(\Delta z)^2}\right],$$

$$\frac{u^{**}_{i,j,k} - u^{n}_{i,j,k}}{\Delta t} = \alpha\left[\frac{1}{2}\frac{\delta^2_x u^{*}_{i,j,k} + \delta^2_x u^{n}_{i,j,k}}{(\Delta x)^2} + \frac{1}{2}\frac{\delta^2_y u^{**}_{i,j,k} + \delta^2_y u^{n}_{i,j,k}}{(\Delta y)^2} + \frac{\delta^2_z u^{n}_{i,j,k}}{(\Delta z)^2}\right],$$

and

$$\frac{u^{n+1}_{i,j,k} - u^{n}_{i,j,k}}{\Delta t} = \alpha\left[\frac{1}{2}\frac{\delta^2_x u^{*}_{i,j,k} + \delta^2_x u^{n}_{i,j,k}}{(\Delta x)^2} + \frac{1}{2}\frac{\delta^2_y u^{**}_{i,j,k} + \delta^2_y u^{n}_{i,j,k}}{(\Delta y)^2} + \frac{1}{2}\frac{\delta^2_z u^{n+1}_{i,j,k} + \delta^2_z u^{n}_{i,j,k}}{(\Delta z)^2}\right]$$

3.11 Consistency Analysis of the Finite Difference Equations

By previous definition, an FDE approximation of a PDE is consistent if the FDE reduces to the original PDE as the step sizes approach zero. In this section, the consistency of some of the methods discussed earlier will be investigated. Since the procedure is simple and straightforward, only a couple of examples are illustrated. As a first example, consider model Equation (3-1), i.e.,

$$\frac{\partial u}{\partial t} = \alpha\frac{\partial^2 u}{\partial x^2}$$

where the FDE approximation by the FTCS explicit method is

$$\frac{u^{n+1}_i - u^n_i}{\Delta t} = \alpha\frac{u^n_{i+1} - 2u^n_i + u^n_{i-1}}{(\Delta x)^2} \tag{3-36}$$

Expand each u in a Taylor series expansion about u^n_i; therefore,

$$u^{n+1}_i = u^n_i + \frac{\partial u}{\partial t}(\Delta t) + \frac{\partial^2 u}{\partial t^2}\frac{(\Delta t)^2}{2!} + O(\Delta t)^3 \tag{3-37}$$

$$u^n_{i+1} = u^n_i + \frac{\partial u}{\partial x}(\Delta x) + \frac{\partial^2 u}{\partial x^2}\frac{(\Delta x)^2}{2!} + \frac{\partial^3 u}{\partial x^3}\frac{(\Delta x)^3}{3!} + O(\Delta x)^4 \tag{3-38}$$

and

$$u^n_{i-1} = u^n_i - \frac{\partial u}{\partial x}(\Delta x) + \frac{\partial^2 u}{\partial x^2}\frac{(\Delta x)^2}{2!} - \frac{\partial^3 u}{\partial x^3}\frac{(\Delta x)^3}{3!} + O(\Delta x)^4 \tag{3-39}$$